Bayesian Estimation of the Expected Mean Square Rate of Repeated Measurements Model

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Abstract: In this paper, we consider the estimators corresponding to the expected mean square rate of repeated measurements model depending on Bayes estimation using Jeffreys' non-informative prior and proper Bayes estimation, and obtaining 14 cases that were classified into five types.

Keywords: Repeated Measurements Model, Bayes Estimation, Jeffreys' Non-Informative Prior, Proper Bayes Estimation.

1. Introduction

Repeated measurements, which are done many times, are observations of the same property. What characterizes such observations from those in the more conventional modelling of statistical data is that the same variable is measured more than once on the same observational unit. As in the study of natural regression, the answers are not independent and more than one observational unit is used. The responses do not constitute a simple time sequence, [1], [2],[10],[11],[15].

In the Bayesian methods to inference, the unknown quantities are viewed as random variables in a probability model for the observed data. Specifically, the set yet undefined parameters are interpreted in the Bayesian method as random variables. Bayesian methods focus on the Monte Carlo Markov chain include what we conclude is the most satisfactory solution to the adaptation of model structure and the path the model is most likely to go in the future. [5] ,[6] ,[7] ,[8] ,[9] ,[11] ,[12], [14].

Many studies have explored the repeated measurement model. for example: Vonesh and Chinchilli (1997) discussed the univariate repeated measurements model , analysis of variance model,[15]. Al-Mouel (2004) studied the multivariate repeated measures models and comparison of estimators, [1]. Al-Mouel and Wang (2004) they studied the asymptotic expansion of the sphericity test for the one-way multivariate repeated measurements analysis of the variance model, [2]. Yin and et al, in (2016), they introduce a Bayesian procedure for the mixed-effects analysis of efficiency studies using mixed binomial regression models subjects in either one- or two-factor repeated-measures designs,[16]. Mohaisen and Khawla in (2016), they introduce a Bayesian procedure for the mixed-effects analysis of efficiency studies using mixed binomial regression models subjects in either one- or two-factor repeated-measures designs, [13]. AL-Mouel, Mohaisen and Khawla in (2017), they are used Bayesian procedure based on Bayes quadratic unbiased estimator to the linear one - way repeated measurements model, [5]. In this work, we consider the estimators corresponding to the expected mean square rate of repeated measurements model depending on Bayes estimation using Jeffreys' non-informative prior and proper Bayes estimation, and obtaining 14 cases that were classified into five types, which are best linear unbiased estimator (BLUE), excepted the type 5 which is bias.

2.Setting Up The Model

The repeated measurement model can be summarized as following:

$$h_{\mathrm{abc}} = \theta + \mathrm{A}_b + \pi_{a(b)} + \mathrm{B}_c + (\mathrm{AB})_{bc} + \epsilon_{abc} \qquad (1)$$

where

$a = 1, \dots, I$ "is an index for experimental unit within group (b)",
$b = 1, \dots, J$ "is an index for levels of the between-units factor (Group)",
$c = 1, \dots, K$"is an index for levels of the within-units factor (Time)",
h_{abc} : "is the response measurement at time (c) for unit (a) within group (b)",
θ : "is the overall mean",
A_b : "is the added effect for treatment group (b)",
$\pi_{a(b)}$: "is the random effect for due to experimental unit (a) within treatment group (b)",

B_c : "is the added effect for time (c)",
$(AB)_{bc}$: "is the added effect for the group (b) $\times$ time (c) interaction",
ϵ_{abc}: "is the random error on time (c) for unit (a) within group (b)".
For the parameterization to be of full rank, we imposed the following set of conditions:
$\sum_{b=1}^{J} A_b = 0$; $\sum_{c=1}^{K} B_c = 0$; $\sum_{b=1}^{J}(AB)_{bc} = 0$ for each c = 1,…, K;
$\sum_{c=1}^{K}(AB)_{bc} = 0$ for each b = 1,…, J.
and let, the ϵ_{abc} and $\pi_{a(b)}$ are independent with
$\epsilon_{abc}\ i.i.d \sim N(0,\sigma_\epsilon^2)$ and $\pi_{a(b)}\ i.i.d \sim N(0,\sigma_\pi^2)$ (2)
The (ANOVA) table of one – way RMM is:

Table (1): ANOVA table of one-way repeated measurement model

Source of Variation	Degree of Freedom	Sum Square	Mean Square	Expected of Mean Square
Group	$J-1$	SS_A	$\frac{SS_A}{J-1}$	$\frac{IK}{J-1}\sum_{b=1}^{J} A_b^2 + K\sigma_\pi^2+\sigma_\epsilon^2$
Unit (Group)	$J(I-1)$	SS_π	$\frac{SS_\pi}{J(I-1)}$	$K\sigma_\pi^2+\sigma_\epsilon^2$
Time	$K-1$	SS_B	$\frac{SS_B}{K-1}$	$\frac{IJ}{K-1}\sum_{c=1}^{K} B_c^2 + \sigma_\epsilon^2$
Group × Time	$(K-1)(J-1)$	$SS_{A\times B}$	$\frac{SS_{A\times B}}{(K-1)(J-1))}$	$\frac{I}{(K-1)(J-1)}\sum_{a=1}^{I}\sum_{c=1}^{K}(AB)_{bc}^2 + \sigma_\epsilon^2$
Residual	$J(K-1)(I-1)$	SS_ϵ	$\frac{SS_\epsilon}{J(K-1)(I-1)}$	σ_ϵ^2

The sum of squares due to groups, subjects group, time, group × time and residuals are then defined respectively as follows:

$SS_A = IK\sum_{b=1}^{K}(\bar{h}_{.b.} - \bar{h}_{...})^2, SS_\pi = K\sum_{a=1}^{I}\sum_{b=1}^{J}(\bar{h}_{ab.} - \bar{h}_{.b.})^2$
$SS_B = IJ\sum_{c=1}^{K}(\bar{h}_{..c} - \bar{h}_{...})^2, SS_{A\times B} = I\sum_{b=1}^{J}\sum_{c=1}^{K}(\bar{h}_{.bc} - \bar{h}_{.b.} - \bar{h}_{..c} + \bar{h}_{...})^2$
$SS_\epsilon = \sum_{a=1}^{I}\sum_{b=1}^{J}\sum_{c=1}^{K}(\bar{h}_{abc} - \bar{h}_{.bc} - h_{ab.} + \bar{h}_{.b.})^2$
where
$\bar{h}_{...} = \frac{1}{IJK}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} h_{abc}$: the overall mean.
$\bar{h}_{.b.} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{c=1}^{K} y_{abc}$: the mean for group (b).
$\bar{h}_{ab.} = \frac{1}{K}\sum_{c=1}^{K} h_{abc}$: the mean for ath subject within group (b).
$\bar{h}_{..c} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} h_{abc}$: the mean for time (c).
$\bar{h}_{.bc} = \frac{1}{I}\sum_{a=1}^{I} h_{abc}$: the mean for group (b) at time (c).
Let
$\theta_{abc} = \theta + A_b + \pi_{a(b)} + B_c + (AB)_{bc}$ (3)
represent the mean of time (c) for unit (a) within group (b).
and, let
$H = \ell_0\theta + \sum_{b=1}^{J}\ell_b A_b + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_a\ell_b\,\pi_{a(b)} + \sum_{c=1}^{K}\ell_c B_c + \sum_{b=1}^{J}\sum_{c=1}^{k}\ell_b\ell_c(AB)_{bc}$ (4)
an arbitrary linear combination of parameters $\theta, A_1, \dots, A_q, \pi_{1(1)}, \dots, \pi_{I(J)}, B_1, \dots, B_K, (AB)_{11}, \dots, (AB)_{JK}$.
the best linear unbiased estimators (BLUE's) of the estimable parameters $\theta, A_b, \pi_{a(b)}, B_c, (AB)_{bc}$ and θ_{abc} are $\hat{\theta} = \bar{h}_{...}$, $\hat{A}_b = \bar{h}_{.b.} - \bar{h}_{...}$, $\hat{\pi}_{a(b)} = (1-r)(\bar{h}_{ab.} - \bar{h}_{.b.})$, $\hat{B}_c = \bar{h}_{..c} - \bar{h}_{...}$, $\widehat{(AB)}_{bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}$ and $\hat{\theta}_{abc} = (1-r)(\bar{h}_{ab.} - \bar{h}_{.b.}) + \bar{h}_{.bc}$, [4].
from the variance analysis (ANOVA) table, we have that
$E(MS_\pi) = \tau_\pi = K\sigma_\pi^2 + \sigma_\epsilon^2$ (5)
and
$E(MS_\epsilon) = \tau_\epsilon = \sigma_\epsilon^2$ (6)
since, the ANOVA estimators of τ_π and τ_ϵ are
$\hat{\tau}_\epsilon = MS_\epsilon$ and $\hat{\tau}_\pi = MS_\pi$ (7)

The rate of expected mean squares is denote

$$r = \frac{\tau_\epsilon}{\tau_\pi} = \frac{\sigma_\epsilon^2}{K\sigma_\pi^2 + \sigma_\epsilon^2} \tag{8}$$

note that $0 < r \leq 1$ is known iff $\sigma_\epsilon^2/\sigma_\pi^2$ is known. And the corresponding estimator of r is

No.1 $\hat{r} = \frac{MS_\epsilon}{MS_\pi} = \frac{SS_\epsilon}{SS_\pi}\frac{1}{(K-1)}$ (9)

These estimates can be beyond the parameter space. To trim the estimated value of $\hat{r}$ by result (9), we put $r > 1$, thus obtaining the estimator

No.2 $\hat{r} = \min\{\frac{SS_\epsilon}{SS_\pi}\frac{1}{(K-1)}, 1\}$ (10)

the trimmed version, No.1, is the usual ANOVA estimator.

3. Estimation of Bayes using Jeffreys' previous non-informative prior

A prior distribution is suggested in which the position parameters θ, $A_b, B_c, (AB)_{bc}$ are taken to be distributed independently of the expected mean squares τ_π and τ_ϵ.

Using Jeffreys' rule , they arrive at a non-informative prior distribution in which μ, θ, $A_b, B_c, (AB)_{bc}$, $\log(\tau_\epsilon)$ and $\log(\tau_\pi)$ are statistically independent with locally uniform distributions. Thus, the non-informative prior distribution has the p.d.f:

$$\boldsymbol{p}(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi) = \boldsymbol{p_1}(A_b, B_c, (AB)_{bc})\boldsymbol{p_2}(\tau_\epsilon, \tau_\pi) \tag{11}$$

with

$\boldsymbol{p_1}(\theta, A_b, B_c, (AB)_{bc}) \propto$ **a constant**

$\boldsymbol{p_2}(\tau_\epsilon, \tau_\pi) \propto (\tau_\epsilon, \tau_\pi)^{-1}$, $(\tau_\pi \geq \tau_\epsilon > 0)$.

alternatively, in term of $\theta, A_b, B_c, (AB)_{bc}, \sigma_\epsilon^2$ and σ_π^2 the prior p.d.f is

$$\boldsymbol{p}(\theta, A_b, B_c, (AB)_{bc}, \sigma_\epsilon^2, \sigma_\pi^2) = \boldsymbol{p_1}(\theta, A_b, B_c, (AB)_{bc})\, \boldsymbol{p_3}(\sigma_\epsilon^2, \sigma_\pi^2) \propto \sigma_\epsilon^{-2}(\sigma_\epsilon^2 + K\sigma_\pi^2)^{-1} \tag{12}$$

subject to the restrictions $(\tau_\pi \geq \tau_\epsilon > 0)$. The likelihood function is

$$L(\theta, A_b, B_c, (AB)_{bc}, \sigma_\epsilon^2, \sigma_\pi^2|h) = [(2\pi\tau_\pi(J+1)\tau_\pi(K\tau_\epsilon + K\sigma_\pi + \tau_\pi)(J+1)(K\tau_\epsilon + K\sigma_\pi + \tau_\pi)]^{-\frac{1}{2}}\left(\frac{SS_\pi}{\tau_\pi}\right)^{\frac{J(I-1)}{2}-1}\left(\frac{SS_\epsilon}{\tau_\epsilon}\right)^{\frac{J(I-1)(K-1)}{2}-1}\tau_\pi^{-1}\tau_\epsilon^{-1}exp\left\{-\frac{1}{2}\left[\frac{IJK(h_{...}-\theta)^2}{\tau_\pi} + \frac{IJK(h_{.b.}-h_{...}-A_b)^2}{(J+1)\tau_\pi} + \frac{IJK(h_{..c}-h_{...}-B_c)^2}{(K\tau_\epsilon+K\sigma_\pi+\tau_\pi)} + \frac{IJK(h_{.bc}+h_{...}-h_{.b.}-h_{..c}-(AB)_{bc})^2}{(J+1)(K\tau_\epsilon+K\sigma_\pi+\tau_\pi)} + \frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\}$$

$$L(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi|h) \propto (\tau_\pi)^{-\frac{J(I-1)-4}{2}}(\tau_\epsilon)^{-\frac{J(I-1)(K-1)}{2}}exp\left\{-\frac{1}{2}\left[\frac{IJK(h_{...}-\theta)^2}{\tau_\pi} + \frac{IJK(h_{.b.}-h_{...}-A_b)^2}{\tau_\pi} + \frac{IJK(h_{..c}-h_{...}-B_c)^2}{\tau_\pi} + \frac{IJK(h_{.bc}+h_{...}-h_{.b.}-h_{..c}-(AB)_{bc})^2}{\tau_\pi} + \frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\} \tag{13}$$

from result (11) and prior of result(13), we have

$$\boldsymbol{p_4}(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi|h) \propto L(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi|h)\boldsymbol{p_1}(\theta, A_b, B_c, (AB)_{bc})\boldsymbol{p_2}(\tau_\epsilon, \tau_\pi) \propto (\tau_\pi)^{-\frac{J(I-1)-4}{2}}(\tau_\epsilon)^{-\frac{J(I-1)(K-1)}{2}}(\tau_\epsilon\, \tau_\pi)^{-1}exp\left\{-\frac{1}{2}\left[\frac{IJK(h_{...}-\theta)^2}{\tau_\pi} + \frac{IJK(h_{.b.}-h_{...}-A_b)^2}{\tau_\pi} + \frac{IJK(h_{..c}-h_{...}-B_c)^2}{\tau_\pi} + \frac{IJK(h_{.bc}+h_{...}-h_{.b.}-h_{..c}-(AB)_{bc})^2}{\tau_\pi} + \frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\}$$

$$\boldsymbol{p_4}(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi|h) \propto (\tau_\pi)^{-\left[\frac{J(I-1)}{2}+3\right]}(\tau_\epsilon)^{-\left[\frac{J(I-1)(K-1)}{2}+1\right]}exp\left\{-\frac{1}{2}\left[\frac{IJK(h_{...}-\theta)^2}{\tau_\pi} + \frac{IJK(h_{.b.}-h_{...}-A_b)^2}{\tau_\pi} + \frac{IJK(h_{..c}-h_{...}-B_c)^2}{\tau_\pi} + \frac{IJK(h_{.bc}+h_{...}-h_{.b.}-h_{..c}-(AB)_{bc})^2}{\tau_\pi} + \frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\}$$

we have

$$\boldsymbol{p_4}(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi|h) \propto (\tau_\pi)^{-\left[\frac{l_\pi}{2}+3\right]}(\tau_\epsilon)^{-\left[\frac{l_\epsilon}{2}+1\right]}exp\left\{-\frac{1}{2}\left[\frac{IJK(h_{...}-\theta)^2}{\tau_\pi} + \frac{IJK(h_{.b.}-h_{...}-A_b)^2}{\tau_\pi} + \frac{IJK(h_{..c}-h_{...}-B_c)^2}{\tau_\pi} + \frac{IJK(h_{.bc}+h_{...}-h_{.b.}-h_{..c}-(AB)_{bc})^2}{\tau_\pi} + \frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\} \tag{14}$$

subject to the restrictions $(-\infty < \theta, A_b, B_c, (AB)_{bc} < \infty, \tau_\pi \geq \tau_\epsilon > 0)$, where $l_\epsilon = J(I-1)(K-1)$ and $l_\pi = J(I-1)$.

The marginal p.d.f of $(\tau_\epsilon, \tau_\pi)$ is given by

$$\boldsymbol{p_5}\,(\tau_\epsilon, \tau_\pi|\boldsymbol{h}) \propto (\tau_\pi)^{-\left[\frac{l_\pi}{2}+3\right]}(\tau_\epsilon)^{-\left[\frac{l_\epsilon}{2}+1\right]}exp\left\{-\frac{1}{2}\left[\frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\} \tag{15}$$

subject to the restrictions $(-\infty < \theta, A_b, B_c, (AB)_{bc} < \infty, \tau_\pi \geq \tau_\epsilon > 0)$.

the p.d.f. (13) is the product of two inverted gamma p.d.f.'s. thus, if these restrictions are ignored, the of r is:

$$\boldsymbol{p_6}(r|\boldsymbol{h}) = \frac{MS_\pi}{MS_\epsilon}\boldsymbol{p}\left(F_{l_\pi, l_\epsilon} = \frac{MS_\pi}{MS_\epsilon}r\right), \quad 0 < r < 1, \tag{16}$$

where $$\boldsymbol{p}(F_{l_\pi, l_\epsilon} = c) = \frac{(\frac{l_\pi}{l_\epsilon})^{\frac{l_\pi}{2}}(\frac{MS_\pi}{MS_\epsilon}r)^{\frac{(l_\pi-2)}{2}}}{B(\frac{l_\pi}{2}, \frac{l_\epsilon}{2})[1+\left(\frac{l_\pi}{l_\epsilon}\right)c]^{\frac{(l_\pi+l_\epsilon)}{2}}} \quad , \ 0 < c < \infty. \tag{17}$$

denote the F- distribution density with l_π and l_ϵ degree of freedom evaluated at (c). We shall refer to (17) as the untruncated marginal posterior p.d.f. of r.
If $I > 1, J > 0$, the distribution mode with this p.d.f is

No.3 $\hat{r} = \frac{SS_\epsilon}{SS_\pi} \frac{J(I-1)-2}{J(I-1)(K-1)+2}$ (18)

If $J(I-1)(K-1) > 2$, the distribution mean with p.d.f (15) is

No.4 $\hat{r} = \frac{SS_\epsilon}{SS_\pi} \frac{J(I-1)}{J(I-1)(K-1)-2}$ (19)

if $J(I-1)(K-1) \leq 2$, the distribution mean is infinite.
It can be show that, when the restrictions $\tau_\pi \geq \tau_\epsilon > 0$ are taken into account, the marginal posterior p.d.f. of r is

$$\boldsymbol{p_7}(r|\boldsymbol{h}) = \frac{\frac{MS_\pi}{MS_\epsilon} \boldsymbol{p}\left(F_{l_\pi, l_\epsilon} = \frac{MS_\pi}{MS_\epsilon} r\right)}{\boldsymbol{p}\left(F_{l_\pi, l_\epsilon} < \frac{MS_\pi}{MS_\epsilon}\right)} \qquad , 0 < r \leq 1, \tag{20}$$

where F_{l_π, l_ϵ} is a random variable which has distribution is F with l_π and l_ϵ degrees of freedom.
The distribution mode, for $I > 1, J > 0$, is the following truncated version of estimator No.3:

No.5 $\hat{r} = \min\{\frac{SS_\epsilon}{SS_\pi} \frac{J(I-1)-2}{J(I-1)(K-1)+2}, 1\}$ (21)

if $J(I-1)(K-1) > 2$, the truncated marginal posterior distribution mean is:

No.6 $\hat{r} = \frac{SS_\epsilon}{SS_\pi} \frac{J(I-1)}{J(I-1)(K-1)-2} \frac{I_x(u+1,\ v-1)}{I_x(u,\ v)}$ (22)

where $x = \frac{SS_\pi}{(SS_\epsilon + SS_\pi)}, u = \frac{J(I-1)}{2}, v = \frac{J(I-1)(K-1)}{2}$.

$$I_x(u, v) = \frac{\int_0^x t^{u-1}(1-t)^{v-1}}{B(u,v)} dt \tag{23}$$

Indicate the incomplete rate of beta function. If $J(I-1)(K-1) > 2$, There is no mean distribution.
Clearly, estimate No.12 is included in the interval (0. 1], and can be composed as:

$$\hat{r} = \frac{u}{v-1} \frac{1-x}{x} \frac{I_x(u+1,\ v-1)}{I_x(u,\ v)} \tag{24}$$

where u, v and x is defined as (22).

4. Proper Estimation of Bayes

Equation (15) suggests that a convenient proper prior p.d.f. for τ_π and τ_ϵ is:

$$\boldsymbol{p_8}\ (\tau_\epsilon, \tau_\pi|\boldsymbol{h}) \propto (\tau_\pi)^{-\left[\frac{l_\pi^*}{2}+3\right]}(\tau_\epsilon)^{-\left[\frac{l_\epsilon^*}{2}+1\right]} exp\left\{-\frac{1}{2}\left[\frac{SS_\pi^*}{\tau_\pi} + \frac{SS_\epsilon^*}{\tau_\epsilon}\right]\right\} \tag{25}$$

Subject to the restrictions $\tau_\pi \geq \tau_\epsilon > 0$.
where $l_\epsilon^*, l_\pi^*, SS_\pi^*$ and SS_ϵ^* are arbitrary positive constants. The likelihood of a set of contrasts of linearly independent error is:

$$L(\tau_\epsilon, \tau_\pi|SS_\pi,\ SS_\epsilon) = (\tau_\pi)^{-1}(\tau_\epsilon)^{-1}\left(\frac{SS_\pi}{\tau_\pi}\right)^{\frac{l_\pi}{2}-1}\left(\frac{SS_\epsilon}{\tau_\epsilon}\right)^{\frac{l_\epsilon}{2}-1} exp\left\{-\frac{1}{2}\left[\frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\}$$

$,(\tau_\pi \geq \tau_\epsilon > 0)$ (26)

$\boldsymbol{p_9}(\tau_\pi, \tau_\epsilon) = L(\tau_\epsilon, \tau_\pi|SS_\pi,\ SS_\epsilon)\boldsymbol{p_4}\ (\tau_\pi, \tau_\epsilon)$

We have

$$\boldsymbol{p_9}(\tau_\pi, \tau_\epsilon) \propto (\tau_\pi)^{-\left[\frac{l_\pi+l_\pi^*}{2}+3\right]}(\tau_\epsilon)^{-\left[\frac{l_\epsilon+l_\epsilon^*}{2}+1\right]} exp\left\{-\frac{1}{2}\left[\frac{SS_\pi+SS_\pi^*}{\tau_\pi} + \frac{SS_\epsilon+SS_\epsilon^*}{\tau_\epsilon}\right]\right\}, (\tau_\pi \geq \tau_\epsilon > 0). \tag{27}$$

If $J(I-1) + l_\pi^* > 2$, the untruncated marginal posterior distribution mode of r is:

No.7 $\hat{r} = \frac{SS_\epsilon+SS_\epsilon^*}{SS_\pi+SS_\pi^*} \frac{J(J-1)+l_\pi^*-2}{J(I-1)(K-1)+l_\epsilon^*+2}$ (28)

and the untruncated marginal posterior distribution mode is:

No.8 $\hat{r} = \min\{\frac{SS_\epsilon+SS_\epsilon^*}{SS_\pi+SS_\pi^*} \frac{J(J-1)+l_\pi^*-2}{J(I-1)(K-1)+l_\epsilon^*+2}, 1\}$ (29)

if $J(I-1)(K-1) + l_\epsilon^* > 2$, the untruncated marginal posterior distribution mean is:

No.9 $\hat{r} = \frac{SS_\epsilon+SS_\epsilon^*}{SS_\pi+SS_\pi^*} \frac{J(I-1)+l_\pi^*}{J(I-1)(K-1)+l_\epsilon^*-2}$ (30)

and the truncated marginal posterior distribution mean is:

No.10 $\hat{r} = \frac{u}{v-1} \frac{1-x}{x} \frac{I_x(u+1,\ v-1)}{I_x(u,\ v)}$ (31)

with

$$x = \frac{SS_\pi+SS_\pi^*}{(SS_\epsilon+SS_\epsilon^*+SS_\pi+SS_\pi^*)},\ u = \frac{J(I-1)+l_\pi^*}{2} = \frac{(l_\pi+l_\pi^*)}{2}\ \ and\ \ v = \frac{J(I-1)(K-1)+l_\epsilon^*}{2} = \frac{l_\epsilon+l_\epsilon^*}{2}.$$

estimators No.7 and No.10 were calculated by solving that the prior (26) data are a linear independent error fixed. This procedure is similar to taking the data to be the vector **y** (for inference about r) and taking the prior p.d.f. of the vector h.
$\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon$ and τ_π to be

$$\boldsymbol{p_{10}}(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi) = \boldsymbol{p_1}(\theta, A_b, B_c, (AB)_{bc})\boldsymbol{p_8}(\tau_\epsilon, \tau_\pi), \quad (32)$$

where

$$\boldsymbol{p_1}(, A_b, B_c, (AB)_{bc}) \propto \textbf{constant}, \quad (33)$$

if the right distribution for $\theta, A_b, B_c and\ (AB)_{bc}$ is chosen, it is useful to assume a prior distribution in which the marginal distribution of τ_ϵ and τ_π is given by (25) and the conditional distribution of τ_ϵ and $\tau_\pi$is given by (25) and the conditional distribution of y and z is considered,

$$\left.\begin{array}{l} \theta \sim N\left(\bar{h}^*_{...}, \frac{\tau_\pi}{N^*}\right) \\ A_b \sim \left(\bar{h}^*_{.b.} - \bar{h}^*_{...}, \frac{(J+1)\tau_\pi}{L^*}\right) \\ B_c \sim \left(\bar{h}^*_{..c} - \bar{h}^*_{...}, \frac{K(\sigma_\epsilon^2 + \sigma_\pi^2) + \tau_\pi}{L^*}\right) \\ (AB)_{bc} \sim (\bar{h}^*_{..c} + \bar{h}^*_{...} - \bar{h}^*_{..k} - \bar{h}^*_{.b.}, \frac{(J+1)[K(\sigma_\epsilon^2 + \sigma_\pi^2) + \Lambda\tau_\pi]}{L^*}) \end{array}\right\} \quad (34)$$

where $\bar{h}^*_{...}, \bar{h}^*_{.b.} - \bar{h}^*_{...}, \bar{h}^*_{..c} - \bar{h}^*_{...}\ and\ \bar{h}^*_{..c} + \bar{h}^*_{...} - \bar{h}^*_{..k} - \bar{h}^*_{.b.}$ are arbitrary constants and $L^* = l^*_\pi + l^*_\epsilon + JK$. We get the posterior p.d.f. by integrating this prior with the chance (29).

$$\boldsymbol{p_{11}}(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi | h) = L(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi | h)\boldsymbol{p_{10}}(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi | h)$$

$$\propto (\tau_\pi)^{-\left[\frac{l_\pi + l^*_\pi}{2} + 5\right]}(\tau_\epsilon)^{-\left[\frac{l_\epsilon + l^*_\epsilon}{2} + 1\right]} exp\left\{-\frac{1}{2}\left(\frac{SS_\pi + SS^*_\pi}{\tau_\pi} + \frac{SS_\epsilon + SS^*_\epsilon}{\tau_\epsilon} + K\right)\right\} \quad (35)$$

according to the restrictions $(-\infty < \theta, A_b, B_c, (AB)_{bc} < \infty, \tau_\pi \geq \tau_\epsilon > 0)$.

where

$$K = \left(\frac{L(\bar{h}_{...} - \theta)^2}{\tau_\pi} + \frac{L^*(\bar{h}^*_{...} - \theta)^2}{\tau_\pi}\right) + \left(\frac{L(\bar{h}_{.b.} - \bar{h}_{...} - A_b)^2}{\tau_\pi} + \frac{L^*\left(\bar{h}^*_{.b.} - \bar{h}^*_{...} - A_b\right)^2}{\tau_\pi}\right) + \left(\frac{L(\bar{h}_{..c} - \bar{y}_{...} - B_c)^2}{\tau_\pi} + \frac{L^*(\bar{h}^*_{..c} - \bar{h}^*_{...} - B_c)^2}{\tau_\pi}\right) + \left(\frac{L(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c} - (AB)_{bc})^2}{\tau_\pi} + \frac{L^*\left(\bar{h}^*_{bc} + \bar{h}^*_{...} - \bar{h}^*_{.b.} - \bar{h}^*_{..c} - (AB)_{bc}\right)^2}{\tau_\pi}\right)$$

$$\frac{L}{\tau_\pi}(\bar{h}_{...} - \theta)^2 + \frac{L^*}{\tau_\pi}(\bar{h}^*_{...} - \theta)^2 = \tau_\pi^{-1}\left[L\bar{h}^2_{...} - 2L\bar{h}_{...}\theta + L\theta^2 + L^*\bar{h}^{*2}_{...} - 2L^*\bar{h}^*_{...}\theta + L^*\theta^2\right]$$

$$= \tau_\pi^{-1}\left[L\theta^2 + L^*\theta^2 - 2\left(L\bar{h}_{...} + L^*\bar{h}^*_{...}\right)\theta + \frac{(L\bar{h}_{...} + L^*\bar{h}^*_{...})^2}{L + L^*} + L\bar{h}^2_{...} + L^*\bar{h}^{*2}_{...} - \frac{(L\bar{h}_{...} + L^*\bar{h}^*_{...})^2}{L + L^*}\right]$$

$$= (L + L^*)\tau_\pi^{-1}\left(\theta^2 - \frac{2(Lh_{...} + L^*\bar{h}^*_{...})\theta}{L + L^*} + \frac{(Lh_{...} + L^*\bar{h}^*_{...})^2}{(L + L^*)^2}\right) + \tau_\pi^{-1}\left(L\bar{h}^2_{...} + L^*\bar{h}^{*2}_{...} - \frac{(L\bar{h}_{...} + L^*\bar{h}^*_{...})^2}{L + L^*}\right)$$

$$= (L + L^*)\tau_\pi^{-1}\left(\theta - \frac{\left(L\bar{h}_{...} + L^*\bar{h}^*_{...}\right)}{L + L^*}\right)^2 + \tau_\pi^{-1}\left(L\bar{h}^2_{...} + L^*\bar{h}^{*2}_{...} - \frac{\left(L\bar{h}_{...} + L^*\bar{h}^*_{...}\right)^2}{L + L^*}\right)$$

$$K = (L + L^*)\tau_\pi^{-1}\left\{\left(\theta - \frac{(L\bar{h}_{...} + L^*\bar{h}^*_{...})}{L + L^*}\right)^2 + \left(\tau_j - \frac{L(\bar{h}_{.b.} - \bar{h}_{...}) + L^*\left(\bar{h}^*_{.j.} - \bar{h}^*_{...}\right)}{L + L^*}\right)^2 + \left(B_c - \frac{L(\bar{h}_{..c} - \bar{h}_{...}) + L^*(\bar{h}^*_{..c} - \bar{h}^*_{...})}{L + L^*}\right)^2 + \left((AB)_{bc} - \frac{L\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c} - (\tau\gamma)_{jk}\right) + L^*\left(\bar{h}^*_{bc} + \bar{h}^*_{...} - \bar{h}^*_{.b.} - \bar{h}^*_{..c} - (AB)_{bc}\right)}{L + L^*}\right)^2\right\} + \tau_\pi^{-1}G$$

$$G = LL^*[\tau_\pi(L + L^*)]^{-1}\left\{\left(\bar{h}_{...} - \bar{h}^*_{...}\right)^2 + \left(\left(\bar{h}_{.b.} - \bar{h}_{...}\right) - \left(\bar{h}^*_{.b.} - \bar{h}^*_{...}\right)\right)^2 + \left(\left(\bar{h}_{..c} - \bar{h}_{...}\right) - \left(\bar{h}^*_{..c} - \bar{h}^*_{...}\right)\right)^2 + \left(\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c} - (AB)_{bc}\right) - \left(\bar{h}^*_{.bc} + \bar{h}^*_{...} - \bar{h}^*_{.b.} - \bar{h}^*_{..c} - (AB)_{bc}\right)\right)^2\right\} \quad (36)$$

and $L = IJK$

the posterior distribution (35) can be rewritten as

$$\boldsymbol{p_{11}}(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi |\ \boldsymbol{h}) = \boldsymbol{p_{12}}(\theta, A_b, B_c, (AB)_{bc} |\ \tau_\epsilon, \tau_\pi, \boldsymbol{h})\boldsymbol{p_{13}}(\tau_\epsilon, \tau_\pi |\ h) \quad (37)$$

where

$$\boldsymbol{p_{12}}(\theta, A_b, B_c, (AB)_{bc} |\ \tau_\epsilon, \tau_\pi, \boldsymbol{h}) \propto \tau_\pi^{-2} exp\left\{-\frac{1}{2}(L + L^*)\tau_\pi^{-1}\left[\left(\theta - \frac{(L\bar{h}_{...} + L^*h)}{L + L^*}\right)^2 + \left(A_b - \frac{L(\bar{h}_{.b.} - \bar{h}_{...}) + L^*(\bar{h}^*_{.b.} - \bar{h}^*_{...})}{L + L^*}\right)^2 + \left(B_c - \frac{L(\bar{h}_{..c} - \bar{h}_{...}) + L^*(\bar{h}^*_{..c} - \bar{h}^*_{...})}{L + L^*}\right)^2 + \left((AB)_{bc} - \frac{L(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c} - (AB)_{bc}) + L^*(\bar{h}^*_{.bc} + \bar{h}^*_{...} - \bar{h}^*_{.b.} - \bar{h}^*_{..c} - (AB)_{bc})}{L + L^*}\right)^2\right]\right\} \quad (38)$$

where $(-\infty < \theta, A_b, B_c, (AB)_{bc} < \infty)$, and

$$\boldsymbol{p_{13}}(\tau_\epsilon, \tau_\pi | h) \propto (\tau_\pi)^{-\left[\frac{l_\pi + l^*_\pi}{2} + 3\right]}(\tau_\epsilon)^{-\left[\frac{l_\epsilon + l^*_\epsilon}{2} + 1\right]} exp\left\{-\frac{1}{2}\left[\frac{SS_\pi + SS^*_\pi + G}{\tau_\pi} + \frac{SS_\epsilon + SS^*_\epsilon}{\tau_\epsilon}\right]\right\} \quad (39)$$

Subject to the restrictions ($\tau_\pi \geq \tau_\epsilon > 0$).

Note that the marginal posterior p.d.f of (τ_π and τ_ϵ is given by (39) and has the same form as (24). Thus, from (39), The above estimators are similar to the No.7-No.10 estimators:

No.11 $\hat{r} = \frac{SS_\epsilon + SS^*_\epsilon}{SS_\pi + SS^*_\pi + G}\frac{J(I-1) + l^*_\pi - 2}{J(I-1)(K-1) + l^*_\epsilon + 2}$ (40)

where $(J(I-1) + l^*_\pi > 2)$

(the untruncated marginal posterior distribution mode of),

No.12 $\hat{r} = \min\{\frac{SS_\epsilon+SS_\epsilon^*}{SS_\pi+SS_\pi^*+G}\frac{J(I-1)+l_\pi^*-2}{J(I-1)(K-1)+l_\epsilon^*+2}, 1\}$, (41)
(the truncated marginal posterior distribution mode),
No.13 $\hat{r} = \frac{SS_\epsilon+SS_\epsilon^*}{SS_\pi+SS_\pi^*+G}\frac{J(I-1)+l_\pi^*}{J(I-1)(K-1)+l_\epsilon^*-2}$ (42)
where $(J(I-1)(K-1)+l_\epsilon^*>2)$
(the untruncated marginal posterior distribution mean), and
No.14 $\hat{r} = \frac{u}{v-1}\frac{1-x}{x}\frac{I_x(u+1,\ v-1)}{I_x(u,\ v)}$ (43)
where $(J(I-1)(K-1)+l_\epsilon^*>2)$
(the truncated marginal posterior distribution mean), with

$$\left.\begin{aligned} x &= \frac{SS_\pi+SS_\pi^*+G}{(SS_\epsilon+SS_\epsilon^*+SS_\pi+SS_\pi^*+G)} \\ u &= \frac{J(I-1)+l_\pi^*}{2} = \frac{l_\pi+l_\pi^*}{2} \\ v &= \frac{J(I-1)(K-1)+l_\epsilon^*}{2} = \frac{l_\epsilon+l_\epsilon^*}{2} \end{aligned}\right\} \quad (44)$$

The corresponding estimators of $\theta, A_b, \pi_{a(b)}, B_c, (AB)_{bc}, \theta_{abc}$ and $H = \ell_0\theta + \sum_{b=1}^{J}\ell_b A_b + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_a\ell_b\,\pi_{a(b)} + \sum_{c=1}^{K}\ell_c B_c + \sum_{b=1}^{J}\sum_{c=1}^{k}\ell_b\ell_c(AB)_{bc}$
are given by

$$\left.\begin{aligned} \hat{\theta} &= \bar{h}_{...} \\ \hat{\pi}_{a(b)} &= (1-\hat{r})(\bar{h}_{ab.}-\bar{y}_{.b.}) \\ \hat{A}_b &= \bar{h}_{.b.}-\bar{h}_{...} \\ \hat{B}_c &= \bar{h}_{..c}-\bar{h}_{...} \\ \left(\widehat{AB}\right)_{bc} &= \bar{h}_{.bc}+\bar{h}_{...}-\bar{h}_{.b.}-\bar{h}_{..c} \\ \hat{\theta}_{abc} &= \bar{h}_{...} + \hat{\mathrm{A}}_b + \hat{\pi}_{a(b)} + \hat{\mathrm{B}}_c + \left(\widehat{\mathrm{AB}}\right)_{bc} \\ \hat{\theta}_{ijk} &= \bar{h}_{.bc}+\hat{\pi}_{a(b)} \\ \hat{H} = \ell_0\theta + \sum_{b=1}^{J}&\ell_b\left(\bar{h}_{.b.}-\bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.}-\bar{y}_{.b.})\right] \\ &+\sum_{c=1}^{K}\ell_c\left(\bar{h}_{..c}-\bar{h}_{...}\right)+\sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc}+\bar{h}_{...}-\bar{h}_{.b.}-\bar{h}_{..c}\right) \end{aligned}\right\} \quad (45)$$

where $(a=1,\ldots,I;\ b=1,\ldots,J;\ c=1,\ldots,K)$. For estimators No.1-No.10, and by

$$\left.\begin{aligned} \hat{\theta} &= \left(\frac{(L\bar{h}_{...}+L^*\bar{h}_{...}^*)}{(L+L^*)}\right), \\ \hat{\pi}_{a(b)} &= (1-\hat{r})(\bar{h}_{ab.}-\bar{h}_{.b.}),\ (a=1,\ldots,I;\ b=1,\ldots,J) \\ \hat{A}_b &= \frac{(L(\bar{h}_{.b.}-\bar{h}_{...})+L^*(\bar{h}_{.b.}^*-h)}{(L+L^*)} \\ \hat{B}_c &= \frac{(L(\bar{h}_{..c}-\bar{h}_{...})+L^*(\bar{h}_{..c}^*-\bar{h}_{...}^*)}{(L+L^*)} \\ (\widehat{AB})_{bc} &= \frac{L(\bar{h}_{.bc}+\bar{h}_{...}-h_{.b.}-\bar{h}_{..c})+L^*(\bar{h}_{.bc}^*+\bar{h}_{...}^*-\bar{h}_{.b.}^*-\bar{h}_{..c}^*)}{(L+L^*)},\ (L=IJK, L^*=l_\epsilon^*+l_\pi^*+1) \\ \hat{\theta}_{abc} &= \hat{\theta} + \hat{\mathrm{A}}_b + \hat{\pi}_{a(b)} + \hat{\mathrm{B}}_c + \left(\widehat{\mathrm{AB}}\right)_{bc} \\ \hat{H} &= \ell_0\hat{\theta} + \sum_{b=1}^{J}\ell_b\hat{A}_b + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_a\ell_b\,\hat{\pi}_{a(b)} + \sum_{c=1}^{K}\ell_c\hat{B}_c \\ &\quad + \sum_{b=1}^{J}\sum_{c=1}^{k}\ell_b\ell_c(\widehat{AB})_{bc} \end{aligned}\right\} \quad (46)$$

where $(a=1,\ldots,I;\ b=1,\ldots,J;\ c=1,\ldots,K)$. For the estimators No.11-No.14.
There are five types of estimators of $\theta, A_b, \pi_{a(b)}, B_c, (AB)_{bc}, \theta_{abc}$ and H. Notice that all estimators for r based on the data only from a full collection of sufficient statistics SS_π, SS_ϵ and $\bar{h}_{...}$.

Type 1: This type consists of the following estimators:
$\hat{\theta}_1 = \bar{h}_{...}$,
$\hat{\pi}_{1;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.}-\bar{y}_{.b.})$,
$\hat{A}_{1;\,b} = \hat{A}_{1,z\,;\,b} = \bar{h}_{.b.}-\bar{h}_{...}$,
$\hat{B}_{1;\,c} = \hat{B}_{1,z\,;\,c} = \bar{h}_{..c}-\bar{h}_{...}$,
$\left(\widehat{AB}\right)_{1;\,bc} = \left(\widehat{AB}\right)_{1,z;\,bc} = \bar{h}_{.bc}+\bar{h}_{...}-\bar{h}_{.b.}-\bar{h}_{..c}$,
$\hat{\theta}_{1;\,abc} = \hat{\theta}_{1,z;\,abc} = \bar{h}_{...} + \hat{\mathrm{A}}_{1,z;\,b} + \hat{\pi}_{1,z;\,a(b)} + \hat{\mathrm{B}}_{1,z;\,c} + \left(\widehat{\mathrm{AB}}\right)_{1,z;bc}$,
$\hat{\theta}_{1,z;\,abc} = \bar{h}_{.bc}+\hat{\pi}_{1,z;\,a(b)}$,
and

$$\hat{H}_1 = \ell_0\theta + \sum_{b=1}^{J} \ell_b(\bar{h}_{.b.} - \bar{h}_{...}) + \sum_{a=1}^{I}\sum_{b=1}^{J} \ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})\right]$$
$$+ \sum_{c=1}^{K} \ell_c(\bar{h}_{..c} - \bar{h}_{...}) + \sum_{b=1}^{J}\sum_{c=1}^{K} \ell_b\ell_c(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}),$$

with

$\hat{r}_1 = \hat{r}_{1,z} = z\frac{SS_\epsilon}{SS_\pi}$.

where $(a = 1,\dots,I;\ b = 1,\dots,J;\ c = 1,\dots,K)$ and z is an arbitrary positive constant. Let $\widehat{\boldsymbol{\theta}}_{\mathbf{1}} = \widehat{\boldsymbol{\theta}}_{\mathbf{1},\boldsymbol{z}}$ denote the vector of dimensions $n \times 1$ whose a*th* component is $\hat{\theta}_{1,z;a}$.

Type 1 estimators will be called untruncated estimators. This type contains No.1, and No.3.

<u>Type 2</u>: This type consists of estimators as follows:

$\hat{\theta}_2 = \bar{h}_{...}$,

$\hat{\pi}_{2;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})$,

$\hat{A}_{2;\,b} = \hat{A}_{2,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{...}$,

$\hat{B}_{2;\,c} = \hat{B}_{2,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{...}$,

$(\widehat{AB})_{2;\,bc} = (\widehat{AB})_{2,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}$,

$\hat{\theta}_{2;\,abc} = \hat{\theta}_{2,z;\,abc} = \bar{h}_{...} + \hat{\mathrm{A}}_{2,z;\,b} + \hat{\pi}_{2,z;\,a(b)} + \hat{\mathrm{B}}_{2,z;\,c} + (\widehat{\mathrm{AB}})_{2,z;bc}$,

$\hat{\theta}_{2,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{2,z;\,a(b)}$,

and

$$\hat{H}_1 = \ell_0\theta + \sum_{b=1}^{J} \ell_b(\bar{h}_{.b.} - \bar{h}_{...}) + \sum_{a=1}^{I}\sum_{b=1}^{J} \ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})\right]$$
$$+ \sum_{c=1}^{K} \ell_c(\bar{h}_{..c} - \bar{h}_{...}) + \sum_{b=1}^{J}\sum_{c=1}^{K} \ell_b\ell_c(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}),$$

with

$\hat{r}_2 = \hat{r}_{2,z} = \min\{z\frac{SS_\epsilon}{SS_\pi}, 1\}$,

where $(a = 1,\dots,I;\ b = 1,\dots,J;\ c = 1,\dots,K)$ and z is an arbitrary positive constant. Let $\widehat{\boldsymbol{\theta}}_{\mathbf{2}} = \widehat{\boldsymbol{\theta}}_{\mathbf{2},\boldsymbol{z}}$ denote the vector of dimensions $n \times 1$ whose a*th* component is $\hat{\theta}_{2,z;a}$.

Type 2 estimators are known as truncated estimators. This type contains No.2 and No.5.

<u>Type 3</u>: This type consists of estimators as follows:

$\hat{\theta}_3 = \bar{h}_{...}$,

$\hat{\pi}_{3;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})$,

$\hat{A}_{3;\,b} = \hat{A}_{3,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{...}$,

$\hat{B}_{3;\,c} = \hat{B}_{3,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{...}$,

$(\widehat{AB})_{3;\,bc} = (\widehat{AB})_{3,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}$,

$\hat{\theta}_{3;\,abc} = \hat{\theta}_{3,z;\,abc} = \bar{h}_{...} + \hat{\mathrm{A}}_{3,z;\,b} + \hat{\pi}_{3,z;\,a(b)} + \hat{\mathrm{B}}_{3,z;\,c} + (\widehat{\mathrm{AB}})_{3,z;bc}$,

$\hat{\theta}_{3,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{3,z;\,a(b)}$,

and

$$\hat{H}_3 = \ell_0\theta + \sum_{b=1}^{J} \ell_b(\bar{h}_{.b.} - \bar{h}_{...}) + \sum_{a=1}^{I}\sum_{b=1}^{J} \ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})\right]$$
$$+ \sum_{c=1}^{K} \ell_c(\bar{h}_{..c} - \bar{h}_{...}) + \sum_{b=1}^{J}\sum_{c=1}^{K} \ell_b\ell_c(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}),$$

with

$\hat{r}_3 = \hat{r}_{3,z} = \mathrm{f}_3(SS_\pi, SS_\epsilon)$,

where $(a = 1,\dots,I;\ b = 1,\dots,J;\ c = 1,\dots,K)$ and $\mathrm{f}_3(x,y)$ is an arbitrary function of $x, y > 0$. Let $\widehat{\boldsymbol{\theta}}_{\mathbf{3}} = \widehat{\boldsymbol{\theta}}_{\mathbf{3},\boldsymbol{z}}$ denote the vector of dimensions $n \times 1$ whose a*th* component is $\hat{\theta}_{3,z;a}$.

This type contains No.1, No.2, No.3, No.5and No.6, we note that types 1 and 2 are special case of type 3.

<u>Type 4</u>: This type consists of the following estimators:

$\hat{\theta}_3 = \bar{h}_{...}$,

$\hat{\pi}_{3;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})$,

$\hat{A}_{3;\,b} = \hat{A}_{3,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{...}$,

$\hat{B}_{3;\,c} = \hat{B}_{3,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{...}$,

$(\widehat{AB})_{3;\,bc} = (\widehat{AB})_{3,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}$,

$\hat{\theta}_{3;\,abc} = \hat{\theta}_{3,z;\,abc} = \bar{h}_{...} + \hat{\mathrm{A}}_{3,z;\,b} + \hat{\pi}_{3,z;\,a(b)} + \hat{\mathrm{B}}_{3,z;\,c} + (\widehat{\mathrm{AB}})_{3,z;bc}$,

$\hat{\theta}_{3,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{3,z;\,a(b)}$,
and
$$\hat{H}_3 = \ell_0\theta + \sum_{b=1}^{J}\ell_b\left(\bar{h}_{.b.} - \bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})\right] + \sum_{c=1}^{K}\ell_c\left(\bar{h}_{..c} - \bar{h}_{...}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}\right),$$

with
$\hat{r}_4 = \hat{r}_4, z = \mathrm{f}_4\left(\frac{SS_\epsilon}{SS_\pi}\right)$,
where $(i = 1, \dots, n, j = 1, \dots, q, k = 1, \dots, q)$ and $\mathrm{f}_4(x_1, x_2)$ is an arbitrary positive function of $x_1 > 0$ and $x_2 > 0$. Let $\hat{\boldsymbol{\mu}}_{\boldsymbol{4}} = \hat{\boldsymbol{\mu}}_{\boldsymbol{4,z}}$ denote the vector of dimensions $n \times 1$ whose a*th* component is $\hat{\theta}_{4,z;a}$.
Note the special cases of Type 4 estimators are Type 3 estimators. In addition to estimators of type 3, This class is composed of No.1, No.2, No.3, No.5, No.6, No.7, No.8, No.9, No.10 estimators.

Type 5: this type consists of estimators as follows:
$\hat{\theta}_5 = \frac{L\bar{h}_{...} + L^*\bar{h}^*_{...}}{(L+L^*)}$,
$\hat{\pi}_{5;\,a(b)} = \hat{\pi}_{5,z\,;a(b)} = (1-\hat{r}_{5,z})(\bar{h}_{ab.} - \bar{h}_{.b.}),\quad (a = 1, \dots, I;\ b = 1, \dots, J)$
$\hat{A}_{5,z;b} = \frac{L(\bar{h}_{.b.} - \bar{h}_{...}) + L^*(\bar{h}^*_{.b.} - \bar{h}^*_{...})}{(L+L^*)}$,
$\hat{B}_{5,z;c} = \frac{L(\bar{h}_{..c} - \bar{h}_{...}) + L^*(\bar{h}^*_{..c} - \bar{h}^*_{...})}{(L+L^*)}$,
$\left(\widehat{AB}\right)_{5,z;\,bc} = \frac{L(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}) + L^*(\bar{h}^*_{.bc} + \bar{h}^*_{...} - \bar{h}^*_{.c.} - \bar{h}^*_{..c})}{(L+L^*)}, (L = IJK, L^* = l^*_\epsilon + l^*_\pi + 1)$
$$\hat{\theta}_{5;abc} = \frac{L\bar{h}_{...} - L^*\bar{h}^*_{...}}{(L+L^*)} + \frac{L(\bar{h}_{.b.} - \bar{h}_{...}) + L^*(\bar{h}^*_{.b.} - \bar{h}^*_{...})}{(L+L^*)} + \hat{\pi}_{5,z\,;\,a(b)} + \frac{L(\bar{h}_{..c} - \bar{h}_{...}) + L^*(\bar{h}^*_{..c} - \bar{h}^*_{...})}{(L+L^*)} + \frac{L(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}) + L^*(\bar{h}^*_{.bc} + \bar{h}^*_{...} - \bar{h}^*_{.b.} - \bar{h}^*_{..c})}{(L+L^*)}$$
$\hat{\theta}_{5;\,abc} = \frac{L\bar{h}_{bc} - 2L^*\bar{h}^*_{...} + L^*\bar{h}^*_{.bc}}{(L+L^*)} + \hat{\pi}_{5,z\,;\,a(b)}$
and
$$\hat{H}_5 = \ell_0\hat{\theta}_{5;\,abc} + \sum_{b=1}^{J}\ell_b A_{5,z;\,b} + \sum_{a=1}^{I}\sum_{b=1}^{j}\ell_a\ell_b\,\hat{\delta}_{5,z\,;\,a(b)} + \sum_{c=1}^{K}\ell_c\gamma_{5,z;\,c} + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c(\tau\gamma)_{5,z;\,bc}$$
where $(a = 1, \dots, I;\ b = 1, \dots, J;\ c = 1, \dots, K)$ and $\hat{r}_5$ depends (nontrivially) on G as well as SS_ϵ and SS_π, with G is define in (34).
$L = IJK$ and $L^* = l^*_\pi + l^*_\epsilon + JK$. Let $\hat{\boldsymbol{\theta}}_{\boldsymbol{5}} = \hat{\boldsymbol{\theta}}_{\boldsymbol{5,z}}$ denote the vector of dimensions $n \times 1$ whose a*th* component is $\hat{\theta}_{5,z;a}$.
This type contains the estimators No.11, No.12, No.13 and No.14.

4. Estimators Description and Category
A complete list of the estimators of r considered in this paper is as follows:
No.1 $\hat{r} = \frac{MS_\epsilon}{MS_\pi} = \frac{SS_\epsilon}{SS_\pi}\frac{1}{(K-1)}$, $\quad (K > 1)$.
No.2 $\hat{r} = \min\{\frac{SS_\epsilon}{SS_\pi}\frac{1}{(K-1)}, 1\}$
No.3 $\hat{r} = \frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)-2}{J(I-1)(K-1)+2}$
No.4 $\hat{r} = \frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)}{J(I-1)(K-1)-2}$,where $J(I-1)(K-1) > 2$
No.5 $\hat{r} = \min\{\frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)}{J(I-1)(K-1)-2}, 1\}$
No.6 $\hat{r} = \frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)}{J(I-1)(K-1)-2}\frac{I_x(u+1,\ v-1)}{I_x(u,\ v)}$,where $J(I-1)(K-1) > 2$
where $x = \frac{SS_\delta}{(SS_e + SS_\delta)}, u = \frac{q(n-1)}{2}, v = \frac{J(I-1)(K-1)}{2}, I_x(u,v) = \frac{\int_0^x t^{u-1}(1-t)^{v-1}}{B(u,v)}dt.$
No.7 $\hat{r} = \frac{SS_\epsilon + SS^*_\epsilon}{SS_\pi + SS^*_\pi}\frac{J(J-1)+l^*_\pi-2}{J(I-1)(K-1)+l^*_\epsilon+2}$,where $J(J-1) + l^*_\pi > 2$
No.8 $\hat{r} = \min\{\frac{SS_\epsilon + SS^*_\epsilon}{SS_\pi + SS^*_\pi}\frac{J(J-1)+l^*_\pi-2}{J(I-1)(K-1)+l^*_\epsilon+2}, 1\}$
No.9 $\hat{r} = \frac{SS_\epsilon + SS^*_\epsilon}{SS_\pi + SS^*_\pi}\frac{J(I-1)+l^*_\pi}{J(I-1)(K-1)+l^*_\epsilon-2}$,where $J(I-1)(K-1) + l^*_\epsilon > 2$
No.10 $\hat{r} = \frac{u}{v-1}\frac{1-x}{x}\frac{I_x(u+1,\ v-1)}{I_x(u,\ v)}$
with $x = \frac{SS_\pi + SS^*_\pi}{(SS_\epsilon + SS^*_\epsilon + SS_\epsilon + SS^*_\epsilon)}, u = \frac{J(I-1)+l^*_\pi}{2} = \frac{(l_\pi + l^*_\pi)}{2}$ $and\ \ v = \frac{J(I-1)(K-1)+l^*_\epsilon}{2} = \frac{l_\epsilon + l^*_\epsilon}{2}$.
No.11 $\hat{r} = \frac{SS_\epsilon + SS^*_\epsilon}{SS_\pi + SS^*_\pi + G}\frac{J(I-1)+l^*_\pi-2}{J(I-1)(K-1)+l^*_\epsilon+2}$,where $J(I-1) + l^*_\pi > 2$
No.12 $\hat{r} = \min\{\frac{SS_\epsilon + SS^*_\epsilon}{SS_\pi + SS^*_\pi + G}\frac{J(I-1)+l^*_\pi-2}{J(I-1)(K-1)+l^*_\epsilon+2}, 1\}$
No.13 $\hat{r} = \frac{SS_\epsilon + SS^*_\epsilon}{SS_\pi + SS^*_\pi + G}\frac{J(I-1)+l^*_\pi}{J(I-1)(K-1)+l^*_\epsilon-2}$,where $J(I-1)(K-1) + l^*_\epsilon > 2$

No.14 $\hat{r} = \frac{u}{v-1} \frac{1-x}{x} \frac{I_x(u+1,\ v-1)}{I_x(u,\ v)}$,where $J(I-1)(K-1) + l_\epsilon^* > 2$

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